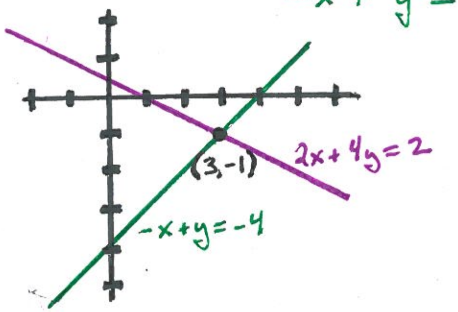
MAT 210 Matrix Multiplication

Def: A matrix is a rectangular array of numbers

EX: $A = \begin{bmatrix} 2 & 4 \\ 1 & -2 \\ 3 & 0 \end{bmatrix}$ $\left\{ \begin{array}{l} A \text{ has } 3 \text{ rows} \\ \qquad\quad 2 \text{ columns} \\ \hookrightarrow \text{a "}3\times 2 \text{ matrix"} \end{array} \right.$

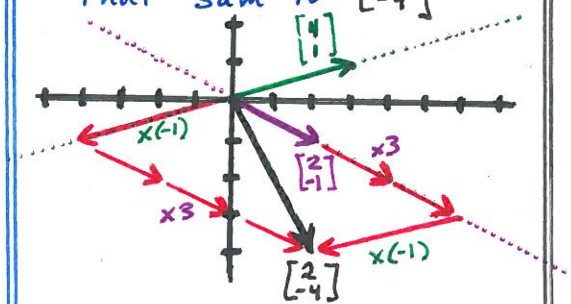
Matrices keep track of equivalent problems:
system of equations & vector equation

EX Find intersection of two lines
$2x + 4y = 2$
$-x + y = -4$



System of Equations
$$\left\{ \begin{array}{l} 2x + 4y = 2 \\ -x + y = -4 \end{array} \right\}$$

Find multiples of vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ that sum to $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$



Vector Equation
$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} 4 \\ 1 \end{bmatrix} y = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Matrix Equation
$$\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Multiplication (Matrix)·(Vector)
is defined so that this correspondence works

EX:
$$\begin{bmatrix} 2 & 4 \\ -1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} x + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} y \quad \text{(as a vector expression)}$$
$$= \begin{bmatrix} 2x + 4y \\ -x \\ 3x + 2y \end{bmatrix} \quad \text{(as a function expression)}$$

EX
$$\begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot 1 + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \cdot 2 + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot 3$$
$$= \begin{bmatrix} 2\cdot 1 + 0\cdot 2 + 1\cdot 3 \\ (-1)\cdot 1 + 3\cdot 2 + (-2)\cdot 3 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

Note: You can only multiply when
length (vector) = # columns (matrix)

Usually we think of multiplication as a collection of dot products: (rows of matrix) • (vector)

$$\begin{bmatrix} \boxed{2 \quad 1} \\ \boxed{3 \quad 4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \boxed{[2 \quad 1]} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \\ \boxed{[3 \quad 4]} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \boxed{2x + y} \\ \boxed{3x + 4y} \end{bmatrix}$$

Multiplication (Matrix)·(Matrix)

Matrix multiplication is a messy operation, which may seem complicated at first, but actually does a very specific, simple job.

Change of Variables

If $A\underline{x} = \underline{y}$ and $B\underline{y} = \underline{z}$

then $\underbrace{(BA)}_{}\underline{x} = B(A\underline{x}) = B\underline{y} = \underline{z}$

product matrix gives $\underline{z}$ in terms of $\underline{x}$

For example, suppose $\begin{bmatrix} 2 & -3 \\ -5 & 4 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ (y in terms of x)

$\begin{bmatrix} 7 & 4 & 2 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ (z in terms of y)

Then the product matrix

$$\underbrace{\begin{bmatrix} 7 & 4 & 2 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -5 & 4 \\ 3 & -2 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 2 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

is $\begin{bmatrix} ?? & ?? \\ ?? & ?? \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ (z in terms of x)

Matrix expressing z_1 & z_2 in terms of x_1 & x_2 $\Longrightarrow \begin{cases} ??\,x_1 + ??\,x_2 = z_1 \\ ??\,x_1 + ??\,x_2 = z_2 \end{cases}$

We can compute the product by converting everything to systems and substituting!

[**Note:** This is a very bad way to compute products; but everyone should do it once in their life.]

$$\begin{bmatrix} 2 & -3 \\ -5 & 4 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightsquigarrow \begin{cases} 2x_1 - 3x_2 = y_1 \\ -5x_1 + 4x_2 = y_2 \\ 3x_1 - 2x_2 = y_3 \end{cases}$$

$$\begin{bmatrix} 7 & 4 & 2 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \rightsquigarrow \begin{cases} 7y_1 + 4y_2 + 2y_3 = z_1 \\ 2y_1 + 3y_2 + 5y_3 = z_2 \end{cases}$$

combine

$$\begin{cases} 7\overbrace{(2x_1 - 3x_2)}^{y_1} + 4\overbrace{(-5x_1 + 4x_2)}^{y_2} + 2\overbrace{(3x_1 - 2x_2)}^{y_3} = z_1 \\ 2(2x_1 - 3x_2) + 3(-5x_1 + 4x_2) + 5(3x_1 - 2x_2) = z_2 \end{cases}$$

factor

$$(*)\quad \begin{cases} (7\cdot 2 + 4(-5) + 2\cdot 3)x_1 + (7(-3) + 4\cdot 4 + 2(-2))x_2 = z_1 \\ (2\cdot 2 + 3(-5) + 5\cdot 3)x_1 + (2(-3) + 3\cdot 4 + 5(-2))x_2 = z_2 \end{cases}$$

simplify

$$\begin{bmatrix} 0 & -9 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \leftsquigarrow \begin{cases} 0x_1 + (-9)x_2 = z_1 \\ 4x_1 + (-4)x_2 = z_2 \end{cases}$$

Product Matrix

So $\begin{bmatrix} 7 & 4 & 2 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -5 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ 4 & -4 \end{bmatrix}$!!

Looking at step (*) in the previous calculation, you should notice a pattern which makes matrix multiplication computable as a series of "dot products"

$$\begin{pmatrix}\underline{\text{row}}\text{ of}\\ \underline{\text{left}}\text{ matrix}\end{pmatrix} \bullet \begin{pmatrix}\underline{\text{column}}\text{ of}\\ \underline{\text{right}}\text{ matrix}\end{pmatrix}$$

$$\begin{bmatrix}\boxed{7 \quad 4 \quad 2}\\ \boxed{2 \quad 3 \quad 5}\end{bmatrix}\begin{bmatrix}\boxed{\begin{matrix}2\\-5\\3\end{matrix}} & \boxed{\begin{matrix}-3\\4\\-2\end{matrix}}\end{bmatrix} = \begin{bmatrix}[7\ 4\ 2]\bullet\begin{bmatrix}2\\-5\\3\end{bmatrix} & [7\ 4\ 2]\bullet\begin{bmatrix}-3\\4\\-2\end{bmatrix}\\ [2\ 3\ 5]\bullet\begin{bmatrix}2\\-5\\3\end{bmatrix} & [2\ 3\ 5]\bullet\begin{bmatrix}-3\\4\\-2\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}14-20+6 & -21+16-4\\ 4-15+15 & -6+12-10\end{bmatrix} = \begin{bmatrix}0 & -9\\ 4 & -4\end{bmatrix}$$

Formula: Multiply matrices by $\begin{pmatrix}\underline{\text{rows}}\text{ of}\\ \underline{\text{left}}\text{ matrix}\end{pmatrix} \bullet \begin{pmatrix}\text{columns of}\\ \underline{\text{right}}\text{ matrix}\end{pmatrix}$

$$B\cdot A = \underbrace{\begin{bmatrix}- \underline{r}_1 -\\ - \underline{r}_2 -\\ \vdots\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}| & | & \\ \underline{c}_1 & \underline{c}_2 & \cdots\\ | & | & \end{bmatrix}}_{A} = \begin{bmatrix}\underline{r}_1\bullet\underline{c}_1 & \underline{r}_1\bullet\underline{c}_2 & \cdots\\ \underline{r}_2\bullet\underline{c}_1 & \underline{r}_2\bullet\underline{c}_2 & \cdots\\ \vdots & \vdots & \end{bmatrix}$$

$$= B\cdot\underbrace{\begin{bmatrix}| & | & \\ \underline{c}_1 & \underline{c}_2 & \cdots\\ | & | & \end{bmatrix}}_{A} = \begin{bmatrix}| & | & \\ B\underline{c}_1 & B\underline{c}_2 & \cdots\\ | & | & \end{bmatrix}$$

(For advanced students: You can get these formulas also by using vector equations instead of systems of equations. Try it!)

EX: $\begin{bmatrix}2 & 3\\ 4 & -1\end{bmatrix}\begin{bmatrix}-3 & 1\\ 7 & 2\end{bmatrix} = \begin{bmatrix}-6+21 & 2+6\\ -12-7 & 4-2\end{bmatrix} = \begin{bmatrix}15 & 8\\ -19 & 2\end{bmatrix}$

EX: $\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 0\end{bmatrix}\begin{bmatrix}-3 & 1\\ 7 & 2\end{bmatrix} = \begin{bmatrix}-3+14 & 1+4\\ -9+28 & 3+8\\ -15+0 & 5+0\end{bmatrix} = \begin{bmatrix}11 & 5\\ 19 & 11\\ -15 & 5\end{bmatrix}$

EX: $\begin{bmatrix}2 & 1 & 3\end{bmatrix}\begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix} = \begin{bmatrix}2+2+9 & 8+5+18\end{bmatrix}$

$= \begin{bmatrix}13 & 31\end{bmatrix}$

Remarks: ① Matrix multiplication is not always defined. To compute $\begin{pmatrix}\text{row of}\\ \text{left matrix}\end{pmatrix}\bullet\begin{pmatrix}\text{column of}\\ \text{right matrix}\end{pmatrix}$ lengths <u>must</u> match

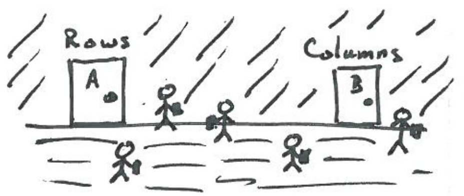
$$\text{length}\begin{pmatrix}\text{row of}\\ \text{left matrix}\end{pmatrix} = \text{length}\begin{pmatrix}\text{column of}\\ \text{right matrix}\end{pmatrix}$$

\# of columns on left = # of rows on right

② Matrix multiplication is not commutative

i.e. $A\cdot B \neq B\cdot A$ (usually)

③ Matrices multiply by <u>column</u> vectors on <u>right</u> and by <u>row</u> vectors on <u>left</u>



"Rows on the left. Columns on the right."

EX: $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 3-4 & 4+10 \\ 9+2 & 12-5 \end{bmatrix} = \begin{bmatrix} -1 & 14 \\ 11 & 7 \end{bmatrix}$

$\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3+12 & 6-4 \\ -2+15 & -4-5 \end{bmatrix} = \begin{bmatrix} 15 & 2 \\ 13 & -9 \end{bmatrix}$

EX: $\begin{bmatrix} \boxed{1 \quad 2} \\ 3 \quad -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 4 & -1 \end{bmatrix}$ Product is Not Defined (lengths do not match)

EX: $\begin{bmatrix} 4 & 1 & -2 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ Product is Not Defined (lengths do not match)

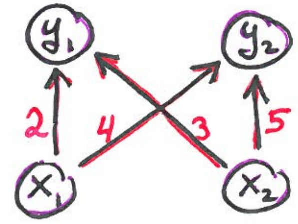
Advanced Note: Instead of thinking of matrices as systems of equations or vector equations we will sometimes want to think of them as (directed) weighted graphs.

For example $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Longleftrightarrow \begin{cases} 2x_1 + 3x_2 = y_1 \\ 4x_1 + 5x_2 = y_2 \end{cases}$

Output is y_1 & y_2 →

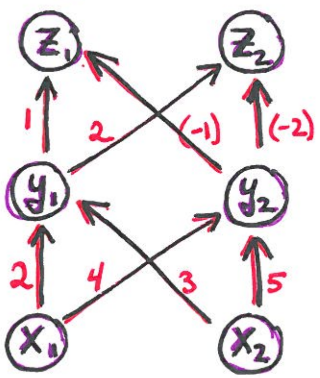
Matrix tells how to combine x's to get y's →

Input is x_1 & x_2 →



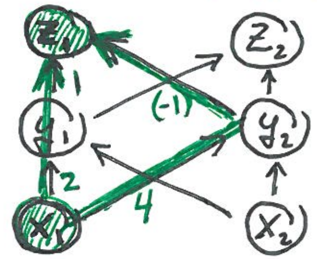
→ Matrix multiplication stacks graphs!

$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\underbrace{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$



Entries in the product matrix trace all paths from an x to a z!

$\begin{bmatrix} 1\cdot 2 + (-1)\cdot 4 & 1\cdot 3 + (-1)5 \\ 2\cdot 2 + (-2)\cdot 4 & 2\cdot 3 + (-2)5 \end{bmatrix}$



This should look familiar...

→ It is exactly how we describe the multivariate chain rule in MAT 120!

Remember: When you multiply by a matrix,

- → the input multiplies each column
- → the output sums each row

input multiplies each column

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$

$$\begin{bmatrix} 2 & 1 & -3 \\ 4 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ -2 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \end{bmatrix} x_3$$

$$\begin{bmatrix} \boxed{2 \quad 1 \quad -3} \\ \boxed{4 \quad -2 \quad 0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \boxed{2x_1 + x_2 - 3x_3} \\ \boxed{4x_1 - 2x_2} \end{bmatrix}$$

output sums each row

So a matrix has a column for each input and a row for each output

$$\begin{bmatrix} 2 & 1 & -3 \\ 4 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Multiply down

$$\begin{array}{cccl} x_1 & x_2 & x_3 & \\ \downarrow & \downarrow & \downarrow & \\ 2 & 1 & -3 & \rightarrow z_1 \\ 4 & -2 & 0 & \rightarrow z_2 \end{array}$$

Add across

$$\left\{ \begin{aligned} 2x_1 + 1x_2 + (-3)x_3 &= z_1 \\ 4x_1 + (-2)x_2 + 0x_3 &= z_2 \end{aligned} \right\}$$

I always imagine that things go in at the top of a matrix and out at the side of the matrix.

Matrices are like undergraduates